*Розглядається задача побудови провідний сенсорної мережі для області складної форми. Для моделювання умов задачі використовуються phi-функції і нові функції приналежності точки області. Будується математична модель спільної задачі покриття і трасування у вигляді задачі нелінійної оптимізації. Пропонуються ефективні алгоритми пошуку локально-оптимальних рішень. Наводяться результати обчислювальних експериментів і порівняння отриманих результатів*

*Ключові слова: кругове покриття, область складної форми, трасування, засоби моделювання, побудова математичної моделі, нелінійна оптимізація*

*Рассматривается задача построения проводной сенсорной сети для области сложной формы. Для моделирования условий задачи используются phi-функции и новые функции принадлежности точки области. Строится математическая модель совместной задачи покрытия и трассировки в виде задачи нелинейной оптимизации. Предлагаются эффективные алгоритмы поиска локально-оптимальных решений. Приводятся результаты вычислительных экспериментов и сравнение полученных результатов.*

*Ключевые слова: круговое покрытие, область сложной формы, трассировка, средства моделирования, построение математической модели, нелинейная оптимизация*

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# CONSTRUCTION OF OPTIMAL WIRE SENSOR NETWORK FOR THE AREA OF COMPLEX SHAPE

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## 1. Introduction

At present, there is a rapidly growing interest in the effective solution of the problems of optimum coverage of areas with geometric objects, a special case of which are the problems of circular coverage. This is explained by a variety of practical applications and extraordinary complexity of mathematical models and methods of their solution.

These problems include the protection of forest arrays from fires, determining the necessary quantity and arrangement of the stations of cellular connection, determining the operation range and arrangement of watering plants, the construction of network, intended for controlling the range of circular orbits of artificial Earth satellites. The problems of selection of the optimum capacities of engine plants of small thrust and theoretical problems of restoring functions, global optimization and construction of optimum quadratures also can be brought to the problems of optimum coverage.

One of the basic fields of application of the problems of coverage, sensory networks, is a relatively new field of research. It is characterized by the rapid development of technologies over recent years. This direction of studies becomes even more relevant, which is proved by the considerably increased intensity of studies in this area [1–3].

The problem of optimum coverage of areas relates to the class of NP– complex ones [4]. Therefore, heuristic methods and algorithms are, as a rule, used for their solution. This indicates the lack of adequate mathematical models and, as a result, the loss of optimal solutions, which considerably narrows the class of the practical problems, which can be effectively solved.

Consequently, developing effective algorithms requires the construction of mathematical models, which are based on the analytical description of relations between the objects in the problem of coverage

Thus, relevant appears the creation of technologies of solving the optimization problems of circular coverage on the basis of structural means of processing information, mathematical and computer simulation and contemporary effective methods of optimization, which will make it possible, based on the source data, to obtain the best coverage option in accordance with the assigned criterion of quality.

## 2. Literature review and problem statement

Many studies are devoted to the problems of construction and analysis of circular coverage [1–3]. One of the basic heuristic approaches to the solution of the problem of circular coverage is the arrangement of sensors in determined units. In this case, the basic template of arrangement is built, which includes several units, with the help of which it is possible to form the division of a plane. If a template is completely covered with zones of sensors, located in its apexes, the entire area can be covered by tessellation of such polygons. The more advanced methods of coverage, based on the arrangement of sensors in determined units, include the arrangement of sensors on the basis of using the Voronoi diagrams [5] and the Delaunay [6] triangulation.

Some researchers use a template in the form of a strip. An effective example of the application of this approach is the method of the sectional-regular coverage of a rectangle with the sections from the circles, placed in the lattice points, proposed in [7].

The second of the basic heuristic approaches to the solution of the problem of circular coverage is based on random arrangement of sensory sensors in the covered region [8]. In this case, in the course of the stochastic construction of coverage, two sub-problems appear: to determine that the coverage is built, and to take away the excessive sensors, the removal of which does not lead to violation of the condition of area coverage. The basic methods of solving these sub-problems are sufficiently highlighted in [1].

One of the most effective is the coverage criterion, which was proposed in [9], based on the analysis of belonging of the points of circles intersection among themselves and with the boundary of the area. If each of such points belongs to the area of servicing of at least one of the other sensors, the coverage has been constructed.

After the random arrangement of sensors, different heuristic optimization methods are frequently used: the method of ants' colony [10], genetic algorithms [11], the method of annealing imitation [12] and others.

While there is a great number of works in this field of research, only some of them are devoted to the construction of precise mathematical models for problems of coverage. In paper [13], mathematical model of the problem on the basis of the Voronoi diagram was constructed. However, the method, proposed in [13], is applicable only to coverage of polygons with identical circles and it requires the introduction of additional variables (apexes of the Voronoi diagrams).

On the basis of the coverage criterion, proposed in [9], the mathematical model of the problem of coverage of an arbitrary polygon with circles of different radii was developed in [14]. The formalization of conditions of coverage with circles of different radii required the introduction of additional variables, which led to a considerable increase in the dimensionality of the problem.

Paper [15] proposed the approach to the coverage of a square with single circles, based on the theory of temperature expansions and compressions of pivotal structures. The basic idea of the algorithm, proposed [16], lies in the construction by the initial system of centers of coverage of Dirichlet-Voronoi and their sequential improvement. The results of the solution of the problem of finding the optimum coverage with the circles of a minimum radius are represented only for a single square and the implementation of the algorithm substantially depends on the shape of the covered region.

Article [17] considers the continuous problem of set coverage as the problem of quasi-differentiated optimization in $R^n$ and presents the algorithm of its solution for the case $R^2$. As in [16], its operation area is restricted by the shape of the covered set.

For the continuous problem of the optimum spherical coverage of a compact set, paper [18] proposed and substantiated the algorithm, based on the use of theory of the optimum division of sets and application of the Shore r-algorithm for solving the obtained problem of optimization of the non-differentiable function. The comparison of the results of this work with the results, given in [13], showed that the probability of finding the optimal solution of the problem decreases with an increase in the dimensionality of a problem.

The shortcomings of the approach, proposed in [18], include the dependence of the results of calculations (magnitude of a minimum radius of spherical coverage) on the parameters of the algorithm: the magnitude of step of three-dimensional grid and magnitude of step of numerical differentiation with the calculation of the components of the generalized gradient.

Article [19] proposes the numerical methods of constructing the coverage, which are based on breaking a set into the Dirichlet areas and finding the so-called characteristic points. The interactive algorithms, proposed in the work, are based on the use of constructions of sub-differential and sub-gradient methods. The special feature of the given modification of the method is the possibility of solving the problem of coverage for arbitrary convex areas with curvilinear boundaries.

Paper [20] contains a mathematical model of the problem of coverage of the convex polygonal area with circles, taking into account the errors in the initial data in the interval form with the use of a class of interval functions; however, the method of its solution is not proposed.

It should be noted, that in articles [13–19], the minimization of radii of the covering sensors is considered as objective function.

Thus, from an analysis of the scientific literature, we should make a conclusion that the overwhelming majority of the works, which refer to the problems of circular coverage, are devoted to studying the heuristic methods of their solution. The analytical models, which have appeared recently, and the methods of solving have a limited area of application or specific shortcomings. This does not make it possible to use them directly for solving the problems of circular coverage of complex regions $R^2$, moreover, to solve the problem of constructing the optimum wire sensory network, to which this study is devoted.

## 3. The aim and tasks of the study

The purpose of present work is the construction of adequate mathematical models and development of effective algorithms of solving the problem of constructing the optimum wire sensory network for area $R^2$. Such algorithms, based on adequate mathematical models, would give the possibility both to search for the local-optimum solutions of the stated problem and to improve the existing solutions, obtained by other methods. In this case, the practical value of the adequately constructed mathematical model lies in the fact that it is applicable for a number of practical problems. Thus, if the problem of coverage can be presented in the form of the problem of nonlinear programming, depending on the criterion of quality, it is possible to increase the reliability of network (to maximize areas of sensors overlapping), to decrease the cost due to minimizing the quantity of sensors or the length of wire connections, etc.

The main tasks of the study are:

– mathematical modeling of coverage restrictions by using phi-functions free from radicals [21] and functions of belonging of the point to the area;

– construction of a mathematical model of the joint problem of coverage of an area by circles of identical radii and trace routing connections in the form of the problem of non-smooth optimization;

– development of methods of the search for approximate and local-optimum solutions of the stated problem.

## 4. Mathematical model of the problem of coverage and the method of its solution

### 4. 1. Construction of mathematical model

Let us assume that we have the assigned closed restricted area $\Omega \subset R^2$ with the piecewise-smooth boundary, formed by L fragments of analytically described curves (for example, by the segments of straight lines and circular arcs), and the set of circles C={$C_i$, i=1,2,…,n}. A number of fragments L may be equal to one (and Ω may be, for example, a circle). Then it is assumed that $C_i=C_i(u_i)=C_i(x_i,y_i)$, point $u_i$ coincides with the center $C_i$. Vector $u_i$ is called a translation vector or a vector of the parameters of the circle $C_i$ arrangement.

Let us construct the union

$$\Upsilon = \bigcup_{i=1}^{n} C_i.$$

The set $\Upsilon$ is called the circular coverage of area Ω, if $\Omega \subseteq \Upsilon$.

Problem setting. To find coverage $\Upsilon$ of area Ω, which is optimum in accordance with a certain criterion of the quality F(u), u=($u_1$, $u_2$,…,$u_n$).

Let us exclude from the examination the coverage, in which there are excessive circles, that is, we assume that

$$\Omega \not\subset \Upsilon \setminus C_i, \forall i \in I_n.$$

The basic methods of detecting excessive circles and some aspects of improvement of the quality of circular coverage are in sufficient detail highlighted in paper [1].

Without disrupting the union for simplification in computations we will not examine the coverage, for which at least one of the boundaries of the connectivity components $\Omega$ does not have common points with the boundaries of circles from $\Upsilon$. If this situation appears, the correspondent component of area Ω is covered with a single circle or the condition of coverage for this component is not satisfied.

Let us form the set P of the points $p_k$ of the boundary of area Ω, in which the smoothness of its boundary is disrupted. Let us designate capacity K of the set P.

Subsequently, we assume that the curvature of the boundary at any point of it, except points from P, is less than the curvature of circles from the set C. The external approximation by the fragments of the curves of smaller curvature is built for the sections of the boundary when this condition is not satisfied.

During the construction of mathematical model of the problem of circular coverage, we used the idea of approach, presented in [14], to the construction of analytical description of circular coverage of the polygonal limited area with the help of the system of phi-functions [21] and functions of belonging of points to areas $R_2$.

Let us take the phi-function $\Phi^{CA}$ of circle C of radius r with the center at point t and an arbitrary object A. Then the function of belonging $\phi^{tA}$ of point t to object A can be defined as function $\Phi^{CA^*}$, A*=R2\int A on condition that r=0. It should be noted that the constructed function is not a phi-function, since the requirement of the agreement of homotopic types of the interior and closure of one of the objects is not satisfied.

Thus, the function of belonging of $\phi^{tA}$ point to set A is called the function, for which it is satisfied: $\phi^{tA} < 0$, if $t \notin A$; $\phi^{tA} = 0$, if $t \in \mathrm{fr}A$; $\phi^{tA} > 0$, if $t \in \mathrm{int}A$.

For example, for circle C of radius r with the center at point ($x_c$, $y_c$), the function of belonging can be represented in the form

$$\phi^{tC} = r^2 - (x_C - x_t)^2 - (y_C - y_t)^2. \quad (1)$$

Let us designate $\Omega^* = R^2 \setminus \mathrm{int}\,\Omega$. It directly follows from the properties of phi-function that:

– if $\Omega = \bigcap_{i}^{m} \Omega_i$, then $\phi^{t\Omega} = \min_{i=1,\dots,m} \phi^{t\Omega_i}$ and

$$\phi^{t\Omega^*} = \max_{i=1,\dots,m} \phi^{t\Omega_i^*}; \quad (2)$$

– if $\Omega = \bigcup_{i=1}^{m} \Omega_i$, then $\phi^{t\Omega} = \max_{i=1,\dots,m} \phi^{t\Omega_i}$ and

$$\phi^{t\Omega^*} = \min_{i=1,\dots,m} \phi^{t\Omega_i^*}. \quad (3)$$

Let us introduce, by analogy with the phi-functions, the normalized function of belonging $\bar{\phi}^{tA}$, for which

$$\phi^{tA} = -\mathrm{dist}(t, A),$$

is satisfied, if $t \notin A$ and

$$\phi^{tA} = \mathrm{dist}(t, A^*),$$

if $t \in A$.

For the pseudonormalized function of belonging $\phi_+^{tA}$, it is correct that

$$\phi_+^{tA} < 0, \text{ if } \mathrm{dist}(t, A^*) < \rho;$$

$$\phi_+^{tA} = 0, \text{ if } \mathrm{dist}(t, A^*) = \rho;$$

$$\phi_+^{tA} > 0, \text{ if } \mathrm{dist}(t, A^*) > \rho.$$

Here ρ>0 is minimally permissible distance,

$$\mathrm{dist}(t, A) = \min_{p \in A} \mathrm{dist}(t, p),$$

$$\mathrm{dist}(t, A^*) = \min_{p \in A^*} \mathrm{dist}(t, p)$$

and

$$\mathrm{dist}(t, p) = \sqrt{(x_t - x_p)^2 + (y_t - y_p)^2}.$$

The example of the normalized function of belonging is function

$$\bar{\phi}^{tC} = r - \sqrt{(x_C - x_t)^2 - (y_C - y_t)^2},$$

and the example of pseudonormalized function of belonging is function

$$\phi_+^{tA} = (r - \rho)^2 - (x_C - x_t)^2 - (y_C - y_t)^2.$$

Normalized and pseudonormalized functions of belonging serve for modeling the conditions of finding point t in area A at the distance to the boundary of the area, not less of the assigned minimally permissible distance ρ.

By analogy with the phi-functions, we will also introduce the function of quasi-belonging $\phi'^{tA}$, depending on the vector of the additional variables $q \in R^g$ and having the property that function $\max_{q\in R^g} \phi'^{tA}$ is the function of belonging. Accordingly, the normalized function of quasi-belonging is called function $\overline{\phi}'^{tA}$, for which $\max_{q\in R^g} \overline{\phi}'^{tA}$ is the normalized function of belonging, and pseudonormalized function of quasi-belonging is called function $\phi_+'^{tA}$, if $\max_{q\in R^g} \phi_+'^{tA}$ is the pseudonormalized function of belonging.

The functions of quasi-belonging make it possible to reformulate some functions of belonging without using the operations of maximum, as well as to considerably simplify writing down other functions of belonging. The compensation for it is the introduction of additional variables and, as a result, an increased dimensionality of the problem.

It should be noted, that for all varieties of the constructed functions, the conditions of the type (2), (3) remain valid.

We will call the circular coverage non-degenerated, if:

– the intersected circles always have a common internal point,

$$C_i \cap C_j \neq \varnothing \to \operatorname{int} C_i \cap \operatorname{int} C_i \neq \varnothing;$$

– none of the points of intersection of circles pairs from the set $\operatorname{fr} C_i, i \in I_n$ belongs to the boundary of area Ω, that is

$$\operatorname{fr} C_i \cap \operatorname{fr} C_j \cap \operatorname{fr} \Omega = \varnothing \quad \forall i, j \in I_n, i \neq j;$$

– no three (and more) circumferences from set $\operatorname{fr} C_i, i \in I_n$ do intersect at one point, that is,

$$\operatorname{fr} C_i \cap \operatorname{fr} C_j \cap \operatorname{fr} C_k = \varnothing \quad \forall i, j, k \in I_n, i \neq j, i \neq k, k \neq j.$$

It should be noted that for the degenerated coverage $\widehat{\Upsilon}$, it is always possible to construct non-degenerated coverage γ, after increasing the radius of each circle forming the coverage by the sufficiently low strictly positive value ε. Here by equivalence we imply the fact that the mathematical model, which describes coverage γ, is also adequate for coverage $\widehat{\Upsilon}$. In this connection we will subsequently examine only non-degenerated coverage. A question of the search for value ε exceeds the scope of this research.

In article [14], the criterion of circular coverage of an arbitrary polygon, which can be generalized in the case of coverage of the arbitrary set Ω is formulated on the basis of ideas from study [9].

For the set

$$\Upsilon = \bigcup_{i=1}^{n} C_i$$

to be the non-degenerate circular coverage of set Ω, it is necessary and sufficient that:

1) for each point $p_k \in P$ at least one circle $C_i, i \in I_n$ should be found so that $p_k \in \operatorname{int} C_i$;

2) for any point

$$t_{ik}^* \in \operatorname{fr} C_i \cap \operatorname{fr} \Omega, i \in I_n, \quad k \in \{1,2\}$$

at least one circle

$$C_{j_k}, j \in I_n, \quad i \neq j,$$

should be found so that $t_{ik}^* \in \operatorname{int} C_{j_k}$ and, consequently, point

$$t_{ijk} \in \operatorname{fr} C_i \cap \operatorname{fr} C_j$$

should belong to $\Omega^*$;

3) for any point

$$t_{ijk} \in \operatorname{fr} C_i \cap \operatorname{fr} C_j, i, j \in I_n, \quad i \neq j, \quad t_{ijk} \in \operatorname{int} \Omega, \quad k \in \{1,2\},$$

there is $C_{s_k}$, $s \neq i$, $s \neq j$, so that $t_{ijk} \in \operatorname{int} C_{s_k}$.

In the process of the construction of mathematical model, satisfaction of the first criterion is provided by the addition to the restriction system of the problem of inequalities having the form $\varphi^{p_k C_i} \geq 0$, of the second criterion – inequalities of the form $\varphi^{t C_j} \geq 0$, of the third criterion – inequalities of the form $\varphi^{t' C_s} \geq 0$.

In [14], criteria 2 and 3 are reformulated and additional variables are introduced in order to avoid the solution of quadratic equations during the calculation of the coordinates of points $t_{ijk}$, as well as to simplify writing down the functions of belonging.

With the use of the circles of an identical radius, the coordinates of the points of form $t_{ijk}$ can be determined analytically on the basis of sufficiently trivial geometric judgments, which makes it possible to formulate the mathematical model of the problem without attracting additional variables.

Paper [14] gives a generalized mathematical model of the problem of coverage of a polygonal set with circles of different radii for solving the problem of optimization of fire-prevention monitoring of forest arrays by taking the relief into account. In this case the radii and the coordinates of the centers of circles are assigned with the help of the splines for modeling the dependence of the radius on the relief of the zones of sensors detection. Relying on the idea of the mathematical model, formulated in [14], let us construct the mathematical model of coverage $n$ by circular zones.

Let us assume that there is the non-degenerated coverage ϒ of area $\Omega$ by circles of an identical radius and it is necessary to optimize a certain quality criterion.

Let us construct the following index sets for coverage ϒ:

– set $\Xi_1$, the elements of which are pairs of numbers: numbers of points from set P and numbers of the circles, which satisfy the conditions of point 1 of the criterion of area coverage;

– set $\Xi_2$, the elements of which are the threes of numbers: numbers of pairs of circles and points of intersection, which satisfy the conditions of point 2 of the criterion of area coverage with circles;

– set $\Xi_3$, the elements of which are the fours of the numbers: numbers of the threes of circles and points of intersection, which satisfy the conditions of point 3 of the criterion of area coverage with circles.

In this case, some excessive elements can be removed from the index sets in the course of construction. For example, the ones that correspond to the pairs of circles, both points of intersection of which belong to the third circle.

Then a sufficiently general mathematical model of the problem of coverage can be written down in the form of

$$\min_{u \in W \subset R^{\delta}} F(u), \tag{4}$$

$$W = \{u \in R^{\delta} : \varphi^{p_k C_i} \geq 0 \forall (k,i) \in \Xi_1,$$

$$\varphi^{t_{ijk}\Omega^*} \geq 0, \Phi_{-}^{C_i C_j} \geq 0 \forall (i,j,k) \in \Xi_2,$$

$$\varphi^{t_{ijk} C_{s_k}} \geq 0, \Phi_{-}^{C_i C_j} \geq 0 \forall (i,j,s) \in \Xi_3,$$

$$k = 1, 2, \Psi \geq 0\}, \qquad (5)$$

where

$\sigma$=2n+l,

u=($u_1$, $u_2$,...$u_n$,t),

$u_i$=($x_i$, $y_i$), i=1, 2, ... , n are the parameters of location of the i-th sensor, $\varphi^{p_k C_i}$, $\varphi^{t_{ik}\Omega^*}$, $\varphi^{t_{ijk} C_{s_k}}$ the functions of belonging of form (1), $t_{ijk}$ is the point of intersection of circumferences $C_i$ and $C_j$, f($u_i$,$u_j$,$u_k$) is the function, calculating coordinates of the points of intersection of circumferences $C_i$ and $C_j$.

$$\Phi_{-}^{C_i C_j} = 4r^2 - (x_i - x_j)^2 - (y_i - y_j)^2,$$

$\Phi_{-}^{C_i C_j}$ is the pseudonormalized phi-function, which formalizes conditions of positioning the pairs of circles at the maximum permissible distance $\rho$=0, t is the vector of auxiliary variables of the problem of the dimensionality l, $\Psi(u)$ is the system of auxiliary restrictions (for example, conditions of belonging of the centers of circles to area $\Omega$).

**4. 2. Study of the properties of the constructed model**

Let us explore the obtained model. In the general case problem (4), (5) is the problem of non-smooth optimization. This is explained by the fact that the function $\varphi^{t_{ijk}\Omega^*}$ in general case is minimax.

In certain cases (for example, when each of the connectivity components of area $\Omega$ is described by one analytical inequality) model (4), (5) describes the classical problem of nonlinear programming.

In other cases (for example, when all components $\Omega$ are convex polygons) model (4), (5) describes the classical problem of nonlinear programming after replacement in (5) of the function of belonging $\varphi^{t_{ijk}\Omega^*}$ with the function of quasi-belonging $\varphi'^{t_{ijk}\Omega^*}$. The compensation in this case is the increased dimensionality of the problem.

If (from the reliability considerations) the maximum permissible distances between the neighboring sensors are assigned, in the model (4), (5) there is an addition of the corresponding pseudonormalized functions.

Let us note that conditions of type $\Phi_{-}^{C_i C_j} \geq 0$ provide the existence of the points of intersection of the selected pairs of objects in the course of solving the problems.

For modeling the problems of constructing the wire sensory network of fire-prevention alarms, the following changes are introduced in model (4), (5):

– objective function is the length of a route (the sum of distances between centers of the sensors, assigned in a certain order);

– conditions of belonging of sensors of an area with consideration of the minimally permissible distances to the area boundary (in the general case they are described with the help of minimax functions), are introduced into the system of additional constraints;

– conditions of the non-belonging of centers of sensors to restricted areas (in the general case they are described with the help of minimax functions) are introduced into the system of additional constraints;

– minimally permissible distances between the centers of sensors (they are described with the help of everywhere smooth phi-function) are introduced into the system of additional constraints.

Taking into account the special features of the mathematical model of the coverage problem, it is proposed to use the strategy of solving the problem, which consists of the following steps:

*Step 1.* We generate the set of the starting points from the area of permissible solutions of problems (4), (5).

*Step 2.* For each of the starting points we construct the model of the form (4), (5).

*Step 3.* We search for the local minimum of objective function F(u) of the problem (4), (5), starting from the points, obtained during step 1, and using the procedure of local optimization with the transformation of the area of permissible solutions, described below.

*Step 4.* We select the best of the local solutions, obtained during step 3 as approximation to the global solution of the problem (4), (5).

**4. 3. Construction of the starting point from the area of permissible solutions**

The construction of the starting point for the problem of designing the wire sensory network is divided into two stages – the construction of coverage and the construction of the route (several routes), connecting the sensors.

For the construction of the starting coverage depending on the special features of the formulation of the problem, we applied the methods of regular coverage on the basis of grids [7], stochastic methods with a change in the coefficient homothety (based on the ideas from [13]) and the optimization by groups of variables, or "consecutive-single coverage". During the application of the last method, the result significantly depends on the used local criterion of the coverage quality. Fairly good results are obtained, if in the calculation of the local criterion, the connectivity of the remained area and the quantity of circles, necessary for covering the remained part of the area in the vicinity of the placed circle, is taken into account.

It should be noted that the constructed starting point often does not belong to the area of feasible solutions of the problem (for example, circles have a larger radius after an increase in homothety coefficient), or it is possible to improve it, by removing some of the circles (this situation frequently appears with the use of regular arrangement). This can be done, after solving auxiliary problems of the form (4), (5). Thus, for instance, for the minimization of a radius of covering circles (identical for all objects), it is possible to accept it as an additional variable and to minimize this variable.

Consideration of technological restrictions is very important with trace routing the wire connections, since two basic forms of wire connections are used: the annular connection, with a large quantity of sensors and the train connection, when several trains with the limited quantity of sensors on each of them can radiate from one point. It is desirable in this case to obtain the minimum length of the wire connections.

If the first problem is a classical problem of a traveling salesman, then the second one can be presented in the form of the modified problem of routing (without returning to the starting point).

For solution of these problems, we used the library of VPRH [22], written in C++, with the open initial code, which implements the set of heuristic methods for solving and optimizing the existing solutions of the problem of routing (and as a special case, the problem of a traveling salesman). The library was thoroughly checked by many researchers on test problems and, on the average, it returns the solution in the limits of 3 % from the optimum. The library of VPRH is easily modified for including additional constraints. Owing to this, the library VPRH is widely used for the solution of practical and scientific problems. Thus, the modification, which makes it possible to select the preferred direction for the routes, was made in this work, which allows an increase in the technological effectiveness of the solution.

**4. 4. Local optimization at the solution of the coverage problem**

In a general case, it is proposed to solve the problem (4), (5) in the form of the sequence of the sub-problems of nonlinear programming.

Even if model (4), (5) describes the problem of nonlinear programming, it can appear that after the process of local optimization, for example, some of the functions $\Phi_-^{C_iC_j} \ge 0$ at the point of local extremum may be equal to zero. This can attest to the fact that the local extremum of the original problem has not been reached and it is necessary to reconstruct the index sets for a new point and to continue the problem solving process.

But if (4), (5) describe the problem of non-differentiable optimization, the area of solution W of the initial problem is represented in the form the union of sub-areas

$$\bigcup_{g=1}^{G} W_g,$$

and one of the sub-areas, which contains the starting point $u^0$, is selected, the search for local extremum on the selected sub-area is carried out starting with the point $u^0$. The point, obtained as a result of solving the sub-problem, is declared as a starting point and the process is repeated until there is some improvement of the objective function.

The search for the local extrema is carried out with the help of the IPOPT program [23].

## 5. Results of computational experiments

**Example 1**. Optimization of trace routing for the problem of coverage the rectangle with dimensions 400×200 with 26 circles of radius 39. The initial coverage is obtained by the regular method, proposed in paper [7]. Improvement reached 7,08 % (Fig. 1).

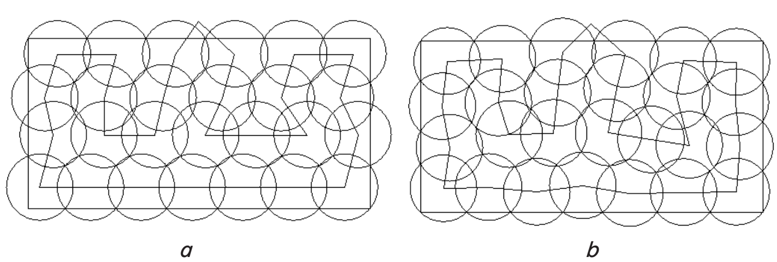



Fig. 1. Result of minimization of length of the route: *a* — 1506.1815 at the starting point; *b* — 1399.4817 after the optimization

The authors did not succeed in finding the papers with results of solving the optimization problems of joint coverage and trace routing. Therefore, we compared the results of the solution of the problem of circular coverage for complex (at least, different from a square and a polygon) areas with the results of one of the latest works [19], as well as with the results of the previous work of this school [24]. The start was at the points, given in articles [19] and [24] as the solution of the problem.

**Example 2**. For the problem of covering the area, limited by the curve $y^2 = x^3 - x$ from paper [19], the following results were obtained:

– for 18 circles – r=0.1621, $u=(u_1,...,u_{18})$=(−0.5258, −0.2435, −0.7620, 0.4522, −0.7660, −0.4290, −0.9404, −0.0209, −0.2750, 0.4364, −0.8342, −0.2391, −0.4015, −0.0462, −0.6984, −0.0305, −0.3022, 0.1310, −0.4951, 0.3276, −0.0930, 0.2027, −0.8658, 0.2360, −0.3308, −0.4487, −0.2136, −0.2531, −0.5709, −0.5384, −0.0882, −0.0719, −0.60710, 0.1765, −0.5333, 0.5829), improvement by 1.8 % (Fig. 2, *a*, *b*);

– for 21 circles – r=0.1467, $u=(u_1,...,u_{18})$=(−0.8733; −0.3312; −0.1751; −0.2438; −0.4475; 0.4962; −0.6831; −0.1288; −0.6587; 0.5467; −0.1344; 0.1760; −0.7289; −0.5483; −0.2967; −0.4273; −0.4992; −0.5138; −0.9643; 0.1699; −0.5870; 0.2807; −0.7513; 0.1212; −0.3784; 0.2265; −0.8176; 0.3865; −0.2904; −0.0201; −0.9130; −0.0908; −0.6290; −0.3372 −0.5261; 0.0118; −0.4295; −0.2349; −0.2244; 0.4016; −0.0281; −0.0469), improvement by 1.6 % (Fig. 2, *c*, *d*).

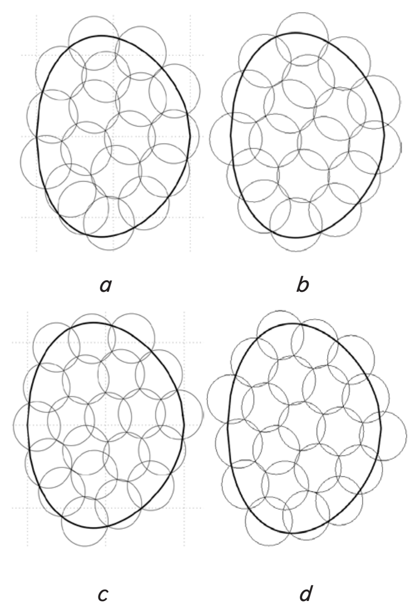


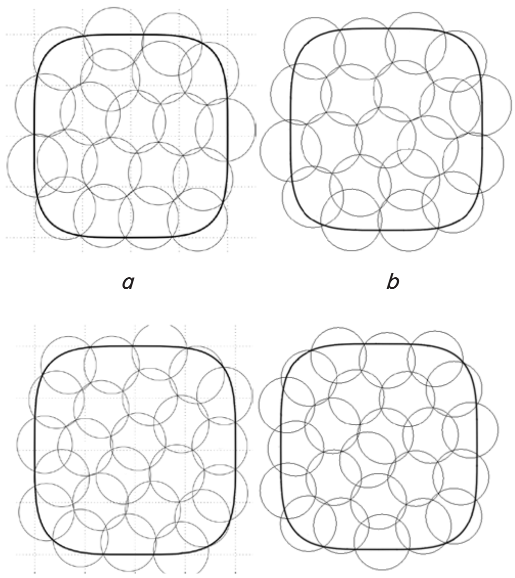

Fig. 2. Comparison of results: *a* — 18 circles from [19]; *b* — 18 circles, obtained in present work; *c* — 21 circles from [19]; *d* — 21 circles, obtained in present work

**Example 3**. For the problems of covering the area, limited by the curve $x^4+y^4=1$ from paper [19], the following results were obtained:

– for 20 circles – r=0.3052, $u=(u_1,...,u_{20})$= =(−0.8397, 0.3582, 0.2301, −0.8929, −0.9424, −0.2013, 0.6705, −0.7140, 0.4200, −0.1630, −0.2498, −0.5479, 0.3021, 0.8639, −0.3403, −0.8896, −0.3411, 0.3520, −0.7034, 0.8137, −0.2082, 0.8077, 0.9484, 0.2500, −0.0037, −0.0698, 0.6412, 0.2200, −0.7227, −0.6623,

−0.5350, −0.1207, 0.8721, −0.327881, 0.6993, 0.6840, 0.1828, 0.3901, 0.2340, −0.4891), improvement by 1,57 % (Fig. 3, *a*, *b*);

– for 24 circles and the area, limited by the curve $x^4+y^4=1$, r=0.2725, $u=(u_1,...,u_{24})$=( −0.9010, 0.0913, −0.3010, 0.4157, 0.1605, 0.5629, 0.8748, 0.5071, 0.2371, 0.9545, −0.7601, −0.2921, −0.2520, 0.8398, −0.4590, 0.0603, −0.7932, −0.6248, −0.0411, −0.1663, 0.1953, −0.5156, −0.6745, 0.7937, 0.4435, −0.2199, 0.6695, −0.6734, 0.4379, 0.2682, 0.8335, 0.0478, −0.3173, −0.4107, −0.7643, 0.4893, −0.5017, −0.8610, 0.622494, 0.7356, 0.8814, −0.4135, −0.0694, −0.8103, 0.0631, 0.1150, 0.3970, −0.9700), improvement by 1.09 % (Fig. 3, *c*, *d*).



*a* *b*

*c* *d*

Fig. 3. Comparison of results: *a* — 18 circles from [19]; *b* — 18 circles, obtained in present work; *c* — 21 circles from [19]; *d* — 21 circles, obtained in present work

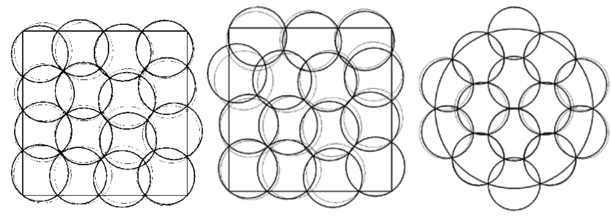
**Example 4**. For the problems from article [24], the following results were obtained:

– for 15 circles and rectangular r=0.3636, $u=(u_1,...,u_{15})$= =(−0.7662, −0.7216, −0.7567, −0.1731, −0.8997, 0.4465, −0.6510, 0.8980, −0.2890, −0.7299, −0.2727, −0.1877, −0.3470, 0.3674, 0.0206, 0.8324, 0.2773, −0.8330, 0.2338, −0.2592, 0.2095, 0.3407, 0.6716, 0.8440, 0.8002, −0.6963, 0.7348, −0.1439, 0.7829, 0.396368), improvement by 0,53 % (Fig. 4, *a*);

– for 16 circles and rectangular r=0.3482, $u=(u_1,...,u_{16})$= =(−0.7882, −0.7236, −0.7494, −0.2055, −0.7178, 0.2402, −0.7902, 0.7221, −0.3258, −0.7583, −0.2554 −0.2779, −0.2574, 0.2764 −0.2982, 0.7960, 0.2070, −0.7960, 0.2817, −0.3344, 0.2664, 0.1547, 0.2155, 0.7399, 0.7447, −0.7634, 0.8391, −0.2178, 0.7505, 0.3339, 0.7236, 0.7885), improvement by 1,1 % (Fig. 4, *b*);

– for 14 circles and circle r=0.3317, $u=(u_1,...,u_{14})$=(0.4539, 0.6354, 0.8423, 0.2924, 0.3886, 0.0002, −0.0006, 0.9430, −0.4546, 0.6348, −0.0002, 0.3267, 0.0002, −0.3267, −0.8426, 0.2915, −0.3886, −0.0002, −0.8423, −0.2925, −0.4539, −0.6354, 0.0006, −0.9430, 0.4547, −0.6348, 0.8426, −0.2915), improvement by 0,23 % (Fig. 4, *c*).

Thus, the performed computational experiments clearly demonstrated the effectiveness of the proposed approach.



*a* *b* *c*

Fig. 4. Comparison of results of minimization of the radius of covering circles (shown by the bold line) with results [24]; *a* — for the problem of coverage with 15 circles; *b* — for the problem of coverage with 16 circles; *c* — coverage of a circle of a single radius with 14 circles

## 6. Discussion of the results of exploring the joint task of coverage and trace routing

It is established that the problem of constructing the optimum wire sensory network for complex areas can be posed in the form of the problem of joint circular coverage and trace routing of connections. Such problems appear, for example, during the construction of the network of sensors of fire-prevention alert in buildings.

Because the problems of circular coverage are difficult to formalize, it was only recently that adequate mathematical models for this class of problems began to appear. However, all the examined models and methods cannot be directly used for the solution of the examined problem (at least, without essential modifications), since:

– there is no possibility of accounting additional technological restrictions for them;

– there is no possibility of covering arbitrary areas;

– rather awkward computational implementation.

In the proposed model all inequalities, which describe the area of permissible solutions of the problem, are assigned by sufficiently simple functions of two types. The first ones serve for analytical description of the conditions of intersections and non-intersections of circles among themselves, the second ones serve for analytical description of belonging of points to sub-areas $R^2$. In the course of solving the problem there is no need for rather awkward computational procedures like the construction of the Voronoi diagrams or the Delaunay triangulation.

Since trace routing in the mathematical model of the problem exists exceptionally in the form of objective function, the represented approach can be easily adapted for the solution of other problems of circular coverage. In particular, the problem of the minimization of radii of covering circles for areas with the curvilinear boundary was solved.

The shortcomings of this model include poor convergence in the course of optimization with the start from an arbitrary point. However, this disadvantage is overcome by the construction of permissible starting points with the use of heuristic methods.

The advantages of the developed method of the search for local-optimum solutions of the problem include the possibility of its use for an improvement in the solutions, obtained by other researchers. In this case, the time consumption at starting from "sufficiently good" points (close to the local extrema of the problem) is rather low. Thus, for solving each of the tasks, represented in chapter 5, time consumption comprised less than 1 second for AMD Athlon(tm) 64x2 Dual Core Processor 5200+.

This work is advancing the studies, carried out in [14] for the problems of covering polygonal areas with the circles of different radii. Subsequently, the development of the represented model for the problems of constructing the optimum wire sensory network with the existence of prohibited areas for routing is planned.

## 7. Conclusions

1. New functions for modeling relationships between geometric objects in the problems of circular coverage were proposed: pseudonormalized functions of belonging of points to areas, functions of quasi-belonging of points to areas, normalized functions of quasi-belonging of points to areas and pseudonormalized functions of quasi-belonging of points to areas.

2. The use of the developed means of mathematical modeling together with phi-functions and pseudonormalized phi-functions made it possible to formalize the description of the restriction of the problem of covering complex areas with identical circles. As a result, it appeared possible to present the problem of constructing the optimum wire sensory network for complex areas in the form of the problem of non-smooth optimization. The lengths of wire connections are used as objective function.

3. In the course of the conducted study, the strategy of the solution and the algorithm of searching for local-optimum solutions for the stated coverage problem were developed. They make it possible to reduce the solution of the appearing problem of non-smooth optimization to the solution of the sequence of sub-problems on the areas of permissible solutions, described by the systems of inequalities with smooth functions. The search for the local extremum begins from the permissible starting points. The construction of starting points is performed within two stages. At the first stage, the heuristic methods of the coverage generation are used, at the second stage we used trace routing of wire connections. The heuristic methods of constructing circular coverage of the areas of complex shape, based on the use of the transformation of coefficients of homothety of circles and on the application of a method of optimization by the groups of variables.

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# FUZZY MODELS OF ROUGH MATHEMATICS

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***Показано, що введений відомий формальний опис неточних множин може бути інтерпретований у термінах нечітких множин. Це дозволяє для розв'язання багатьох задач неточної математики використати розвинений апарат нечіткої математики. Наведено приклад розв'язання задачі лінійного програмування, параметри якої визначені неточно. Для опису неточних параметрів задачі використані функції (L-R)-типу. Для розв'язання задачі введено складений критерій. Чисельне значення критерію враховує міру близькості отримуваного результату до модального рішення і рівень компактності функції приналежності значення цільової функції***

***Ключові слова: неточна математика, нечіткі моделі неточних чисел, рішення задач неточної математики, неточне лінійне програмування***

***Показано, что введённое известное формальное описание неточных множеств может быть интерпретировано в терминах нечётких множеств. Это позволяет для решения многих задач неточной математики использовать развитый аппарат нечёткой математики. Приведён пример решения задачи линейного программирования, параметры которой определены неточно. Для описания неточных параметров задачи использованы функции (L-R)-типа. При решении задачи введён составной критерий. Численное значение критерия учитывает меру близости получаемого результата к модальному решению и уровень компактности функции принадлежности получаемого значения целевой функции***

***Ключевые слова: неточная математика, нечёткие модели неточных чисел, решение задач неточной математики, неточное линейное программирование***

## 1. Introduction

The practical problems of analyzing and synthesizing complex systems are solved under conditions of uncertainty. The degree of uncertainty is determined by the level of knowledge on the state and behavior of a system under study and the environment in which the system operates. It is essential to take this uncertainty into account while solving problems of assessing and predicting the states of systems in engineering [1, 2], military affairs [3, 4], medicine [5], economy [6, 7], as well as problems of structural and parametric optimization [8–10].

Over the past few decades, the emergence and rapid development of the fuzzy set theory [11–21] have significantly expanded the range of tasks for which it has become possible to use a strict formal mathematical apparatus. The presence of a rod element in this theory means a fuzzy value membership function as a natural analogue of the distribution